2-D MATH OPERATIONS

CONTINUOUS 2-D FUNCTIONS

2-D CONVOLUTION

Definition

- Extension of 1-D convolution

$$g(x, y) = f(x, y) \circledast \circledast h(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha, \beta)h(x-\alpha, y-\beta)d\alpha d\beta$$

Properties

- generally same as 1-D convolution (commutative, distributive, associative)

Define commutative, distributive, associative operations

Simplifies for separable functions

- **General case** $g(x, y) = f(x, y) \circledast\circledast h(x, y)$

- **Case I: h (or f) separable**

$$g(x, y) = f(x, y) \circledast\circledast [h_1(x)h_2(y)]$$
$$= f(x, y) \circledast h_1(x) \circledast h_2(y)$$

two 1-D convolutions

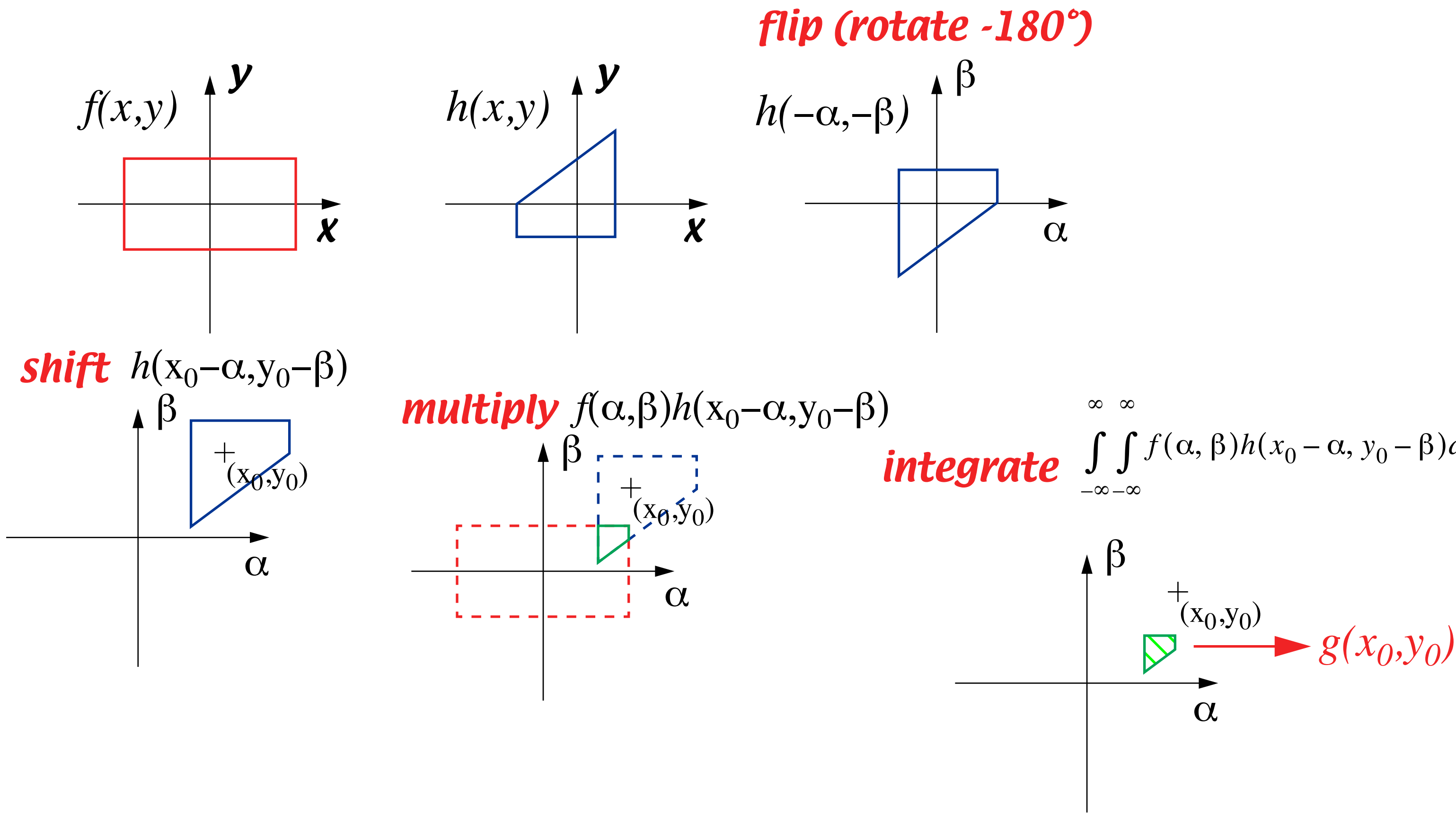
- ***Case II: f and h separable***

$$g(x, y) = [f_1(x) * h_1(x)][f_2(y) * h_2(y)]$$
$$= g_1(x)g_2(y)$$

g(x,y) is also separable

Example

- *For simple graphics, let f and h be binary "mask" functions, equal to one within the boundary and zero elsewhere*



$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)h(x_0-\alpha,y_0-\beta)\,d\alpha\,d\beta$$

2-D CORRELATION

Identical to 2-D convolution, except neither function is "flipped"

$$g(x, y) = f(x, y) \star\star\, h(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha, \beta)h(\alpha - x, \beta - y)d\alpha d\beta$$

Important difference

- *not* **commutative**

$$f \star\star\, h \neq h \star\star\, f$$

2-D MATH OPERATIONS

2-D FOURIER TRANSFORMS

Forward transform $$F(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{-j2\pi(xu + yv)}dxdy$$

Inverse transform $$f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v)e^{j2\pi(xu + yv)}dudv$$

Properties

- ***1-D transform properties generally also apply to 2-D transform***
- ***Useful property:***

$$F(u, v) = \int_{-\infty}^{\infty} e^{-j2\pi yv}\left(\int_{-\infty}^{\infty} f(x, y)e^{-j2\pi xu}dx\right)dy$$

2-D Fourier Transform of Separable Functions

$$f(x, y) = f_1(x) f_2(y)$$

$$F_1(u) = \int_{-\infty}^{\infty} f_1(x) e^{-j2\pi xu} dx \ , \quad F_2(v) = \int_{-\infty}^{\infty} f_2(y) e^{-j2\pi yv} dy$$

$$F(u, v) = \int_{-\infty}^{\infty} e^{-j2\pi yv} \left(\int_{-\infty}^{\infty} f(x, y) e^{-j2\pi xu} dx \right) dy$$

$$= \int_{-\infty}^{\infty} f_2(y) e^{-j2\pi yv} \left(\int_{-\infty}^{\infty} f_1(x) e^{-j2\pi xu} dx \right) dy = F_1(u) \int_{-\infty}^{\infty} f_2(y) e^{-j2\pi yv} dy$$

$$= F_1(u) F_2(v)$$

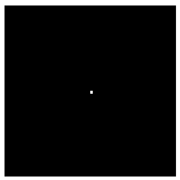
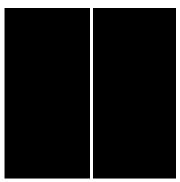
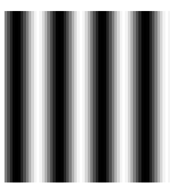
- **Product of two 1-D tranforms**
- **Note the difference from convolution theorem!**

2-D Fourier Transform Properties

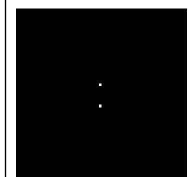
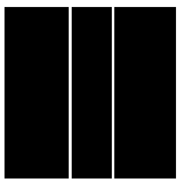
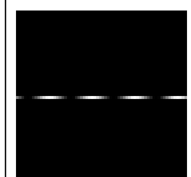
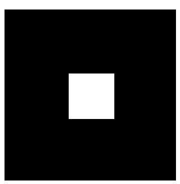
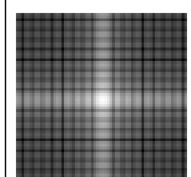
name	f(x,y)	F(u,v)
separability	$f_1(x)f_2(y)$	$F_1(u)F_2(v)$
scaling	$f(x/a, y/b)$	$\|a\|\|b\|F(au, bv)$
shifting	$f(x \pm a, y \pm b)$	$e^{\pm j2\pi(au+bv)}F(u, v)$
linearity (superposition)	$af_1(x, y) + bf_2(x, y)$	$aF_1(u, v) + bF_2(u, v)$
convolution	$f_1(x, y)**f_2(x, y)$	$F_1(u, v) \cdot F_2(u, v)$
	$f_1(x, y) \cdot f_2(x, y)$	$F_1(u, v)**F_2(u, v)$
affine	$f(ax+by+c, dx+ey+f)$	$\frac{1}{\|\Delta\|}e^{-j\left(\frac{2\pi}{\Delta}\right)[(bf-ec)u+(dc-af)v]}F\left(\frac{eu-dv}{\Delta}, \frac{-bu+av}{\Delta}\right)$, where $\Delta = ae - db$

Show that the affine property reduces to the shifting property or the scaling property with appropriate parameters

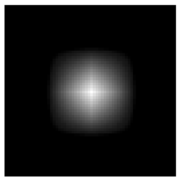
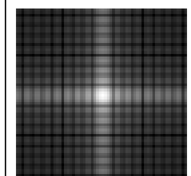
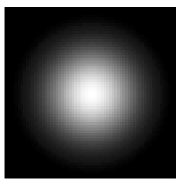
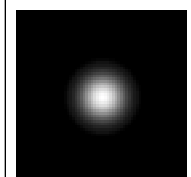
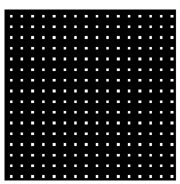
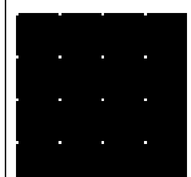
2-D Fourier Transform Pairs

descriptive name of f(x,y)	picture of f(x,y)	f(x,y)	F(u,v)	descriptive name of F(u,v)	picture of |F(u,v)|
delta		$\delta(x, y)$	1	***constant***	
blade		$\delta(x)$	$\delta(v)$	***blade***	
ripple		$\cos(2\pi ax)$	$\frac{1}{2\|a\|}\delta\delta\left(\frac{u}{a}\right)\delta(v)$	***double delta***	

2-D MATH OPERATIONS

descriptive name of f(x,y)	picture of f(x,y)	f(x,y)	F(u,v)	descriptive name of F(u,v)	picture of |F(u,v)|
ripple		$\cos(2\pi a y)$	$\frac{1}{2\|a\|}\delta\delta\left(\frac{v}{a}\right)\delta(u)$	**double delta**	
double blade		$\frac{1}{2\|a\|}\delta\delta\left(\frac{x}{a}\right)$	$\cos(2\pi a u)\delta(v)$		
box		$rect(x, y)$	$\mathrm{sinc}(u, v)$		

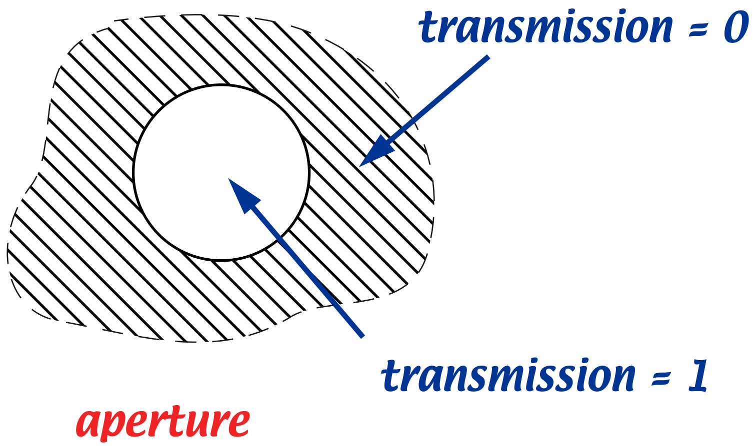
2-D MATH OPERATIONS

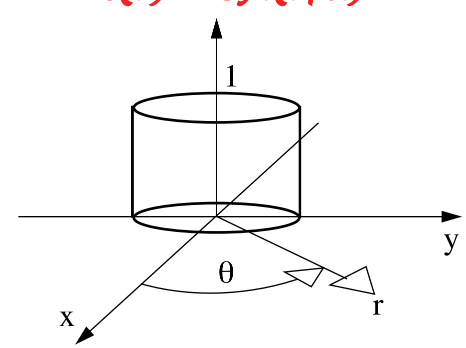
descriptive name of f(x,y)	picture of f(x,y)	f(x,y)	F(u,v)	descriptive name of F(u,v)	picture of |F(u,v)|
		$tri(x, y)$	$\mathrm{sinc}^2(u, v)$		
		$gaus(x, y)$	$gaus(u, v)$		
grid or bed-of-nails		$comb(x, y)$	$comb(u, v)$	**grid or bed-of-nails**	

2-D FOURIER TRANSFORMS IN POLAR COORDINATES

Useful for radial functions

- cyl(r/d) describes the transmission function of a circular optical aperture





Derive a special Fourier transform for such functions

$$F(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(xu + yv)} dx dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\sqrt{x^2 + y^2}) e^{-j2\pi(xu + yv)} dx dy$$

- **Substitute:**

$$x = r\cos\theta \qquad u = \rho\cos\phi$$
$$y = r\sin\theta \qquad v = \rho\sin\phi$$

space domain (for x, y) — *spatial frequency domain* (for u, v)

where

$$r = \sqrt{x^2 + y^2} \qquad \theta = \operatorname{atan}(y/x)$$

and

$$\rho = \sqrt{u^2 + v^2} \qquad \phi = \operatorname{atan}(v/u)$$

Hankel Transform

- ***2-D Fourier transform in polar coordinates***

$$F(\rho\cos\phi, \rho\sin\phi) = \int_0^{\infty} r\,dr \int_0^{2\pi} d\theta f(r) e^{-j2\pi\rho r\cos(\theta-\phi)}$$

$$= \int_0^{\infty} f(r) r\,dr [2\pi J_0(2\pi\rho r)]$$

where $J_0(\,)$ is the zero-order Bessel function of the first kind.

- ***Simplifying, we have the zero-order Hankel transform,***

$$F(\rho\cos\phi, \rho\sin\phi) = 2\pi\int_0^{\infty} f(r) J_0(2\pi\rho r) r\,dr$$

- ***If $f(r,\theta) = f(r)$ (circularly symmetric), then $F(\rho,\phi) = F(\rho)$***

Inverse Hankel transform

$$f(r) = 2\pi\int_0^{\infty} F(\rho)J_0(2\pi r\rho)\rho d\rho$$

Hankel Transform Properties

name	f(r)	F(ρ)
scaling	$f\left(\frac{r}{b}\right)$	$\lvert b\rvert^2 F(b\rho)$
convolution	$f(r) \circledast\circledast\, h(r)$	$F(\rho)H(\rho)$

Hankel Transform Pairs

$f(r)$	$F(\rho)$
$\frac{\delta(r)}{\pi r}$	1
$1/r$	$1/\rho$
$cyl(r)$	$\frac{\pi}{4} somb(\rho)$
e^{-r}	$\frac{2\pi}{(4\pi^2\rho^2 + 1)^{3/2}}$
$gaus(r)$	$gaus(\rho)$

- ***Summary - continuous 2-D functions***

case	*convolution*	*Fourier transform*
general	$f(x, y) \circledast \circledast h(x, y)$	$F(u, v)H(u, v)$
h separable	$f(x, y) \circledast h_1(x) \circledast h_2(y)$	$F(u, v)H_1(u)H_2(v)$
both separable	$[f_1(x) \circledast h_1(x)][f_2(y) \circledast h_2(y)]$	$F_1(u)H_1(u)F_2(v)H_2(v)$
radial functions	$f(r) \circledast \circledast h(r)$	$F(\rho)H(\rho)$

DISCRETE 2-D FUNCTIONS

2-D CONVOLUTION

Definition

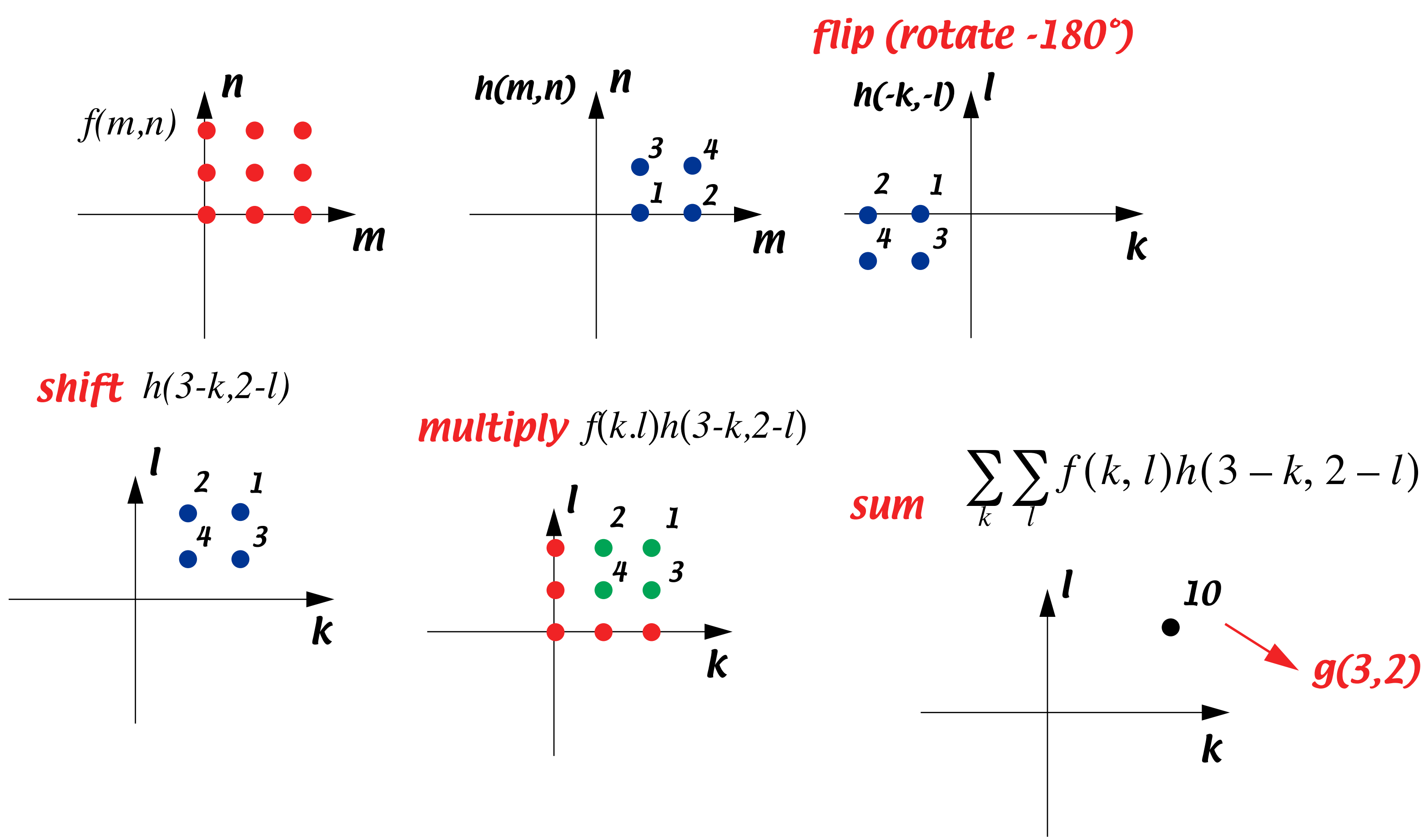
- *Extension of 1-D convolution*

$$g(m,n) = f(m,n) \circledast \circledast h(m,n) = \sum_k \sum_l f(k,l) h(m-k, n-l)$$

Properties

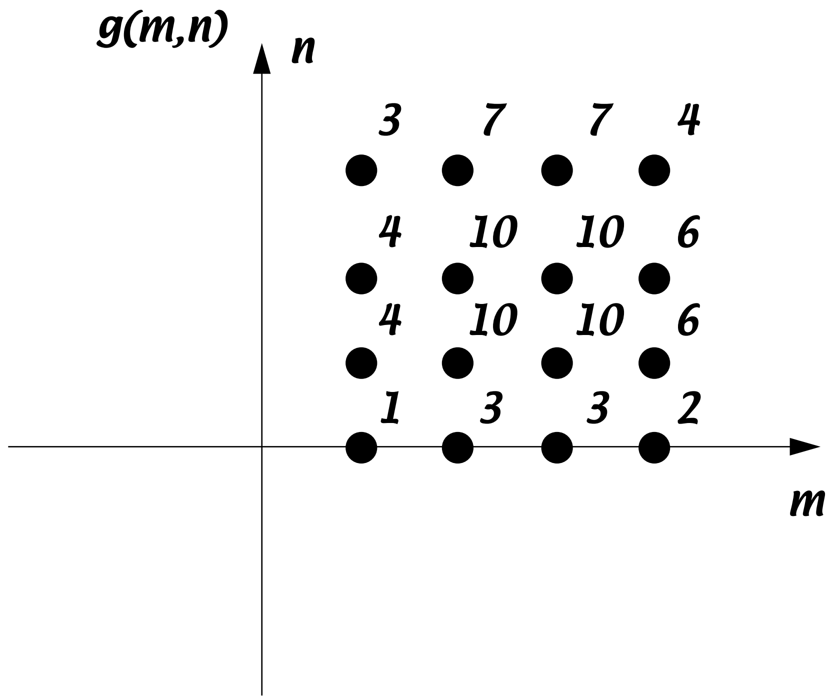
- *generally same as 1-D convolution (commutative, distributive, associative)*

Example



$$\sum_k \sum_l f(k, l)h(3-k, 2-l)$$

Final result



- If f is $M \times N$ and h is $K \times L$, then g is $(M + K - 1) \times (N + L - 1)$
- Note phase shift in m in $g(m,n)$ due to linear phase in $h(m,n)$

Convolution with separable h

$$
\begin{aligned}
g(m,n) &= \sum_k \sum_l f(k,l)h(m-k,n-l) \\
&= \sum_k \sum_l f(k,l)h_1(m-k)h_2(n-l) \\
&= \sum_k h_1(m-k) \sum_l f(k,l)h_2(n-l)
\end{aligned}
$$

- ***inner sum is 1-D convolution between f and h_2***

  array size –> M x (N + L - 1)

- ***outer sum is 1-D convolution between result of inner sum and h_1***

  array size –> (M + K - 1) x (N + L - 1)

For N x N image and M x M filter, show that computational advantage of the separable filter is M/2