Achievable Information Rates and the Coding-Spreading Tradeoff in Finite-Sized Synchronous CDMA Systems

Zuqiang Tang and William E. Ryan
ECE Dept., The University of Arizona
P.O. Box 210104, Tucson, AZ 85721
{tangzq,ryan}@ece.arizona.edu

***Abstract*— In this paper, we compute achievable rates for synchronous code-division multiple-access (CDMA) systems and study the associated coding-spreading tradeoff problem using these results. We assume random spreading sequences and the computed achievable rates are averaged over ensembles of spreading sequences. Unlike most prior work which analyzed the spectral efficiency of large CDMA systems under Gaussianity assumptions (inputs and multi-user interference), we make no such assumptions. In order to display the coding-spreading tradeoff, we plot the required minimum SNR for reliable transmission as a function of information rate. It is shown that the coding-spreading tradeoff favors all coding (i.e., no spreading) when the optimal multiuser detector (MUD) is employed, whereas for systems with suboptimal MUDs and single-user decoding, there generally exist optimal code rates. We also provide simulation results on the performance of LDPC-coded synchronous CDMA systems which approach the information-theoretic limits we have computed.**

I. Introduction

Direct-sequence code-division multiple-access (CDMA) systems have been extensively studied in the last two decades. Its ability to support more mobile users than time-division multiple-access (TDMA) and frequency-division multiple-access (FDMA) mobile wireless systems has been widely acknowledged. Hence, CDMA may be considered an enabling technology in present and future wireless systems.

In recent years, some important work has been reported on the information-theoretic channel capacity of CDMA systems. References [1], [2], [3], [4] provide analysis on the spectral efficiency of large (i.e, large number of users) CDMA systems on additive white Gaussian noise (AWGN) and flat-fading channels. Reference [5] gives an insightful new approach to the analysis of the capacity of large CDMA systems. However, the assumptions underlying large CDMA systems, namely, the Gaussianity of multiple access interference (MAI), may not always hold in real systems. Other approaches include the assumption of Gaussian channel inputs in capacity problems and the adoption of spherical spreading sequences [6], [7], which would facilitate the analysis of the systems.

In another area, recent developments in channel coding have yielded codes capable of near-capacity operation on many channels, including the binary symmetric channel (BSC), the binary erasure channel (BEC), the additive white Gaussian noise (AWGN) channel, and fading channels. These capacity-approaching codes include turbo, turbo-like, and low-density parity-check (LDPC) codes [8], [9], [10]. It has been more difficult to determine whether or not these codes are capable of near-capacity performance on binary-input interference channels such as MAI channels in part because it is difficult to compute the capacity of these channels (except for some special cases and under different assumptions which may not be realistic).

It is natural to ask how the capacity of binary-input CDMA (BI-CDMA) systems with random spreading may be achieved with channel coding. A number of researchers have presented analysis and simulation results in the literature to demonstrate the efficacy of channel codes on CDMA channels [11], [12], [13], [14] and the possibility of their approaching the information-theoretic capacity of CDMA systems. In view of the constraints on bandwidth in modern communication systems, it is also natural to consider the tradeoff in bandwidth expansion due to spreading and that due to coding. References [15], [16] have presented results on this tradeoff problem for large CDMA systems. However, because of the large CDMA assumptions, these works also adopt the Gaussian-input and Gaussian-MAI assumptions.

In this paper, we consider small CDMA systems where the Gaussian assumptions do not hold and we constrain the inputs to be binary random variables as is true in actual systems. The achievable information rates we compute may be considered to be binary-input capacity or constrained capacity. Lastly, we consider the capacity to be dependent on the receiver and so we consider a number of receivers (MUDs), from the optimal multi-user receiver to the decorrelating receiver.

The rest of the paper is outlined as follows. Section II presents the synchronous CDMA models adopted in this work. Section III provides the analysis of the binary-input capacity of CDMA systems with different MUDs. Section IV provides numerical results for the capacities of small CDMA systems and gives some LDPC simulation results. Section V contains conclusions.

II. Synchronous CDMA System Model

In this paper, the baseband model for a K-user BPSK-modulated synchronous CDMA system is adopted. The received CDMA signal can be written as

$$\mathbf{y} = \sum_{i=1}^{K} a_i x_i \mathbf{s}_i + \mathbf{w} \tag{1}$$

where a_i, x_i, and $\mathbf{s}_i$ are, respectively, the amplitude, the binary (code) bit, and the normalized spread vector (signature sequence) of the i^{th} user; $\mathbf{w}$ is the channel AWGN vector with mean $\mathbf{0}$ and covariance $\sigma^2\mathbf{I}$. All vectors, $\mathbf{y}$, $\mathbf{s}_i$, and $\mathbf{w}$, are $N \times 1$ vectors, where N is the length of spreading sequences.

The discrete-time equivalent model for the synchronous CDMA system at the output of the match-filter bank (see Fig. 1) is

$$\mathbf{Y} = \mathbf{RAX} + \mathbf{n} \tag{2}$$

where $\mathbf{Y}$ is the $K \times 1$ vector output of the matched filter bank. $\mathbf{R}$ is the $K \times K$ correlation matrix of the spreading sequences of the K users,

$$\mathbf{R} = [\mathbf{s_1} \ \mathbf{s_2} \ \cdots \ \mathbf{s}_K]^T [\mathbf{s_1} \ \mathbf{s_2} \ \cdots \ \mathbf{s}_K] \,. \tag{3}$$

$\mathbf{A}$ is a $K \times K$ diagonal matrix of the amplitudes of the K users, which is $A = diag\{a_1 \ a_2 \ \cdots \ a_K\}$. $\mathbf{X} = [x_1 \ x_2 \ \cdots \ x_K]^T$ is the binary-input vector of the K users, $x_i \in \{-1, +1\}$; $\mathbf{n}$ is a $K \times 1$ Gaussian noise vector with mean $\mathbf{0}$ and covariance $\mathbf{R}$, which is the matched-filter bank output of the channel AWGN vector.

To simplify our analysis, we assume all users have the same power at the receiver. Therefore, the matrix $\mathbf{A}$ becomes an identity matrix and can be omitted. Since all users are symmetric under the assumption of random spreading, the ”capacity” (achievable information rate), averaged over the ensemble of spreading sequences, is the same for all K users. Thus, we will consider only the information rate of the first user in the analysis and simulations in this paper. The reason we consider the binary-input achievable information rate for each user, instead of the sum capacity of the entire system, is that it easily leads us to optimal code rates.

Optimality here is in the sense of minimizing the signal-to-noise ratio required for reliable transmission with a given bandwidth expansion factor. The bandwidth expansion factor, Q, of the CDMA system is defined as the ratio of the spread sequence length, N, to the code rate r,

$$Q = N/r. \tag{4}$$

The optimal code rate yields the solution to the coding-spreading tradeoff problem. Heuristically, we imagine that spreading is used to combat MAI and coding is used to combat noise. (However, as described in [17], for the optimal MUD, no spreading should be employed. Our results below support this.)

III. Multiuser Detectors and Their Binary-Input Capacities

Multiuser detectors for CDMA systems have been studied for more than 15 years [18]. Many MUDs, from the optimal MUD to single user matched filter detector, have been proposed with different performances and complexities. In this paper, we will study the binary-input capacities of random-spread synchronous CDMA systems for the following receivers:

- optimal MUD
- BCJR-once MUD
- MMSE MUD
- decorrelating MUD
- matched-filter detector

Descriptions of these MUDs may be found in the literature [18], [19], [20]. Below we present binary-input mutual information rate expressions for each of these.

A. Optimal MUD

By definition, the optimal MUD achieves the largest binary-input capacity (C_{bi}). Given a randomly selected set of spread sequences, the per-user expression for C_{bi} of a synchronous CDMA system with an optimal MUD is given by

$$\begin{aligned} I(\mathbf{Y};\mathbf{X})/K &= [H(\mathbf{Y}) - H(\mathbf{Y}|\mathbf{X})]/K \\ &= [H(\mathbf{Y}) - H(\mathbf{n})]/K \end{aligned} \tag{5}$$

where

$$H(\mathbf{Y}) = -\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{K} p_{\mathbf{Y}}(\mathbf{y}) log(p_{\mathbf{Y}}(\mathbf{y}))\, d\mathbf{y} \tag{6}$$

and

$$H(\mathbf{n}) = \frac{1}{2} log(2\pi e)^K |\mathbf{R}| \tag{7}$$

for positive definite $\mathbf{R}$.

There is no simple closed-form formula for the capacity expression (5) which involves a K-dimensional integration. However, it is possible to obtain numerical results for such C_{bi} in (5) when K is not too large. In the next section, we will use Monte-Carlo integration to obtain values for C_{bi}.

The capacity given in (5) is a function of the randomly chosen spreading sequences and, hence, is a random variable. By averaging (5) over the ensemble of random spreading sequences of length N, we obtain the binary-input capacity for the synchronous CDMA system with an optimal MUD. That is,

$$C^*_{opt} = E_{\{\mathbf{s_1}, \mathbf{s_2}, \cdots, \mathbf{s_K}\}}[I(\mathbf{Y};\mathbf{X})/K] \tag{8}$$

This capacity could be approximately approached by an iterative MUD which employs a BCJR (or BCJR-like) detector in cooperation with a soft-in/soft-out decoder [11], [12], [13], [14]. The BCJR-once MUD, described in the next section, represents a compromise iterative multi-user detector/decoder in which the BCJR-MUD iterates only once.

B. BCJR-once MUD/Decoder

In [20], a low-complexity suboptimal detector/decoder, called the BCJR-once detector/decoder was introduced for application to LDPC-coded ISI channels. A discussion on the similar problem of separation of MUD and coding can also be found in [19]. Here we analyze a suboptimal MUD which applies a similar idea. Unlike the iterative MUD, the BCJR-once MUD performs optimal MUD only once, so that there are no iterations between the MUD and the (single-user) decoders.

In this case, the per-user C_{bi} can be expressed as

$$I(\mathbf{Y}; x_1) \quad = \quad H(\mathbf{Y}) - H(\mathbf{Y}|x_1) \tag{9}$$

where $H(\mathbf{Y})$ is as in (6) and

$$H(\mathbf{Y}|x_1) = H(\mathbf{R}\mathbf{X}' + \mathbf{n}). \tag{10}$$

Here we define $\mathbf{X}' \triangleq [0, x_2, x_3, \cdots, x_K]^T$. Then, the averaged C_{bi} for the synchronous CDMA system with BCJR-once MUD/decoder is,

$$C^*_{BCJR-once} = E_{\{\mathbf{s_1}, \mathbf{s_2}, \cdots, \mathbf{s_K}\}}[I(\mathbf{Y}; x_1)] \tag{11}$$

C. MMSE MUD

The three other MUDs discussed in this paper are linear MUDs: the MMSE MUD, the decorrelating MUD, and the MF single-user detector. Since they are all linear MUDs and share the same mathematical form, their binary-input capacities can be expressed in the same form.

The general form of the linear MUD for one user is

$$\begin{aligned} Z &= \mathbf{L}^T\mathbf{Y} \\ &= \mathbf{L}^T\mathbf{R}\mathbf{X} + \mathbf{L}^T\mathbf{n} \end{aligned} \tag{12}$$

where $\mathbf{L}$ is a $K \times 1$ vector which represents the linear MUD. Therefore, the C_{bi} per user of the synchronous CDMA system with a linear MUD is

$$I(Z; x_1) = H(Z) - H(Z|x_1) \tag{13}$$

where

$$H(Z|x_1) = H(\mathbf{L}^T(\mathbf{R}\mathbf{X}' + \mathbf{n})) \tag{14}$$

and $\mathbf{X}' = [0, x_2, x_3, \cdots, x_K]^T$. Here $H(Z)$ and $H(Z/x_1)$ both involve infinite integrations and can be calculated numerically. The (averaged) binary-input capacity per user is then

$$C^*_{linear} = E_{\{\mathbf{s_1}, \mathbf{s_2}, \cdots, \mathbf{s_K}\}}[I(Z; x_1)] \tag{15}$$

For the MMSE MUD, $\mathbf{L}$ for user 1 is given by [18]

$$\mathbf{L_{MMSE}} = ([1\ 0\ 0\ \cdots\ 0](\mathbf{R} + \sigma^2\mathbf{I})^{-1})^T \tag{16}$$

Therefore, the C_{bi} per user for a fixed random sequence set of the synchronous CDMA system with an MMSE MUD is

$$I(Z_{MMSE}; x_1) = H(Z_{MMSE}) - H(Z_{MMSE}|x_1) \tag{17}$$

where

$$Z_{MMSE} = \mathbf{L_{MMSE}}^T\mathbf{R}\mathbf{X} + \mathbf{L_{MMSE}}^T\mathbf{n} \tag{18}$$

Thus, the averaged the C_{bi} per user is

$$C^*_{MMSE} = E_{\{\mathbf{s_1}, \mathbf{s_2}, \cdots, \mathbf{s_K}\}}[I(Z_{MMSE}; x_1)] \tag{19}$$

D. Decorrelating MUD

The decorrelating MUD is a linear detector as in (12) with $\mathbf{L} = \mathbf{L_{DEC}}$, where

$$\mathbf{L_{DEC}} = ([1\ 0\ 0\ \cdots\ 0]\mathbf{R}^{-1})^T \tag{20}$$

As for the MMSE MUD, we have the expression of the binary-input capacity per user of the synchronous CDMA system with decorrelation MUD,

$$I(Z_{DEC}; x_1) = H(Z_{DEC}) - H(Z_{DEC}|x_1) \tag{21}$$

where

$$Z_{DEC} = \mathbf{L_{DEC}}^T\mathbf{R}\mathbf{X} + \mathbf{L_{DEC}}^T\mathbf{n} \tag{22}$$

so that the averaged C_{bi} per user is

$$C^*_{DEC} = E_{\{\mathbf{s_1}, \mathbf{s_2}, \cdots, \mathbf{s_K}\}}[I(Z_{DEC}; x_1)] \tag{23}$$

The difference between the decorrelating MUD and the MMSE MUD is that decorrelating MUD requires the inverse of the correlation matrix $\mathbf{R}$. Since we are studying random spread CDMA systems, sometimes $\mathbf{R}$ is not invertible and therefore, the decorrelating MUD is not implementable and no information can be transmitted through the channel. In our numerical analysis, we will discard such cases.

E. Matched-Filter Single-User Detector

The MF detector is a linear detector as in (12) with $\mathbf{L} = \mathbf{L_{MF}}$, where

$$\mathbf{L_{MF}} = [1\ 0\ 0\ \cdots\ 0]^T \tag{24}$$

.

We then have the expression of the binary-input capacity per user of synchronous CDMA systems with MF detector,

$$I(Z_{MF}; x_1) = H(Z_{MF}) - H(Z_{MF}|x_1) \tag{25}$$

where

$$Z_{MF} = \mathbf{L_{MF}}^T\mathbf{R}\mathbf{X} + \mathbf{L_{MF}}^T\mathbf{n} \tag{26}$$

.

Averaging over the set of spreading sequences, we obtain

$$C^*_{MF} = E_{\{\mathbf{s_1}, \mathbf{s_2}, \cdots, \mathbf{s_K}\}}[I(Z_{MF}; x_1)] \tag{27}$$

.

IV. Numerical and Simulation Results

In this section, we use Monte Carlo integration [21] to numerically compute the integrals presented in the previous section. We also present some simulation results for LDPC-coded CDMA systems.

Due to the high complexity of multi-dimensional numerical integration, we only consider small synchronous CDMA systems with $K = 4$ users and two different bandwidth expansions factors: $Q = 8$ and $Q = 20$. These correspond to systems spectral efficiencies, $\eta \triangleq \frac{K}{N} r = \frac{K}{Q}$, 0.5 and 0.2.

To obtain the averaged C_{bi} per user, 2000 random spreading sequence sets for the 4 users were generated. The algorithm is as follows(For the decorrelating MUD, since the correlation matrix $\mathbf{R}$ needs to be positive definite, the length of spreading sequences N has to be bigger than or equal to the number of users K).

Algorithm

1) Select and fix the number of users K and the bandwidth expansion factor Q.
2) For $N = 1$ to Q,
   (*For the decorrelating MUD*: For $N = K$ to Q,)
   a) generate K random spreading sequences of length N.
   b) for each MUD, use Monte Carlo integration to calculate C_{bi} of CDMA systems with the formulas given in the previous section.
   c) if haven't done this for 2000 spreading sequence sets, go to step a). Otherwise, continue.
   d) average these 2000 C_{bi}-E_s/N_0 curves to obtain the averaged C_{bi}-E_s/N_0 curve, where $E_s/N_0 = N/Q \times E_b/N_0$.
   e) find the required minimum $(\frac{E_b}{N_0})^*_N$ corresponding to the code rate $r_N = N/Q$. In this case, the code rate r_N is the maximum information rate the CDMA user can achieve with the signal-to-noise ratio $(\frac{E_b}{N_0})^*_N$ and spreading gain N.
3) Plot the $(\frac{E_b}{N_0})^*_N$-r_N curve obtained from Step 2.

This algorithm leads to the results presented in Fig. 2 and Fig. 3.

From both Fig. 2 and Fig. 3, we can see that, for systems employing the optimal MUD, the more the bandwidth given to coding, the less E_b/N_0 that is required. That is, no spreading/all coding achieves the optimal coding-spreading tradeoff. This is reasonable and consistent with the results of [1], [15] for large systems and [17].

However, things are different for suboptimal detectors. In Fig. 2 (corresponding to $Q = 8$), systems with BCJR-once, MF, and MMSE detectors all have optimal code rates in the middle when $K > N$. But the system with decorrelating MUD favors all coding (no spreading).

In Fig. 3 (corresponding to $Q = 20$), we can see that systems with suboptimal detectors except the decorrelating MUD follow the same trend as the system with optimal MUD.

We have also performed simulations to assess the performance of synchronous CDMA systems with iterative multiuser detection/LDPC decoding. We expect this to approach the performance of the optimal joint multiuser detector/decoder whose binary-input capacity is given in the previous section. A spreading sequence set for each user is randomly picked and fixed. The reason we do this is that, from our simulations, only when the correlation matrix $\mathbf{R}$ is positive definite, the achievable information rates vary little with different random spreading sequences. The iterative decoding method is as described in [14] which requires the matrix $\mathbf{R}$ to be positive definite (instead of using convolutional codes, we are using LDPC codes.).

To compare with our capacity results, we choose a 4-user synchronous CDMA system. Two moderate length (8000) irregular LDPC codes of rates 0.5 and 0.8 are used. The bandwidth expansion factor Q is 20. The simulated bit error rates of these systems are shown in Fig. 4. Also included in the figure are: (a) the capacity limits on the synchronous CDMA channel from Fig. 3 and (b) performance curves for these codes on the binary-input AWGN channel. We notice that the effect of the CDMA channel is a performance loss (relative to the AWGN channel) of about 0.4dB for the rate 0.5 code and 0.7dB for the rate 0.8 code. More importantly, at bit error rate 10^{-4}, the performances of the LDPC codes are about 1dB from the capacity curves of Fig. 3.

V. Conclusions

In this paper, the binary-input achievable information rates (capacity) for synchronous CDMA systems with a small number of users is studied. The coding-spreading tradeoff problem in synchronous CDMA systems is also addressed. Numerical results show there do exist optimal code rates for synchronous CDMA systems with certain suboptimal MUDs. However, all coding (no spreading) is optimal for the optimal joint detection/decoding receiver. Simulations of synchronous CDMA systems with iterative optimal multiuser detection and LDPC decoding were also presented to show that the binary-input capacities can be closely approached with practical schemes. Future research would focus on the coding-spreading tradeoff problems with asynchronous CDMA and fading channels. More carefully considered channel coding schemes would also be an interesting topic in this area.

Acknowledgment

The authors acknowledge the help of Michael Yang who designed the irregular LDPC codes and who helped debug the decoder.

References

[1] S. Verdu and S. Shamai, "Spectral efficiency of CDMA with random spreading," *IEEE Trans. Inf. Theory*, vol. 45, Mar. 1999, pp.622-640

[2] P. Rapajic and D. Popescu, "Information capacity for a random signature multiple-input multiple-output channel," *IEEE Trans. Comm.*, vol. 48, no. 8, Aug. 2000, pp. 1245-1248

[3] S. Shamai and S. Verdu, "The impact of frequency-flat fading on the spectral efficiency of CDMA," *IEEE Trans. Info. Theory*, vol. 47, no. 4, May 2001, pp. 1302-1327

[4] R. R. Muller and S. Verdu, "Design and analysis of low-complexity interference mitigation on vector channels," *IEEE JSAC*, vol. 19, no. 8, Aug. 2001, pp. 1429-1441

[5] T. Tanaka, "A statistical-mechanics approach to large-system analysis of CDMA multiuser detectors," *IEEE Trans. Infor. Thory*, vol 48, no. 11, Nov. 2002, pp. 2888-2910

[6] R. R. Muller, "Multiuser receivers for randomly spread signals: fundamental limits with and without decision-feedback," *IEEE Trans. Infor. Theory*, vol. 47, no. 1, Jan. 2002, pp. 268-283

[7] R. R. Muller and W. Gerstacker, "On the capacity loss due to sparation of detection and decoding in large CDMA systems," *IEEE ITW2002*, 2002, pp. 222

[8] C. Berrou, A. Glavieux, and P. Thitimajshima, "Near Shannon limit error-correcting coding and decoding: Turbo codes," *Proc. 1993 Int. Conf. Comm.*, pp. 1064-1070

[9] R. Gallager, "Low-density parity-check codes," *IRE Trans. Infor. Theory*, vol. 8, no. 1, Jan. 1962, pp. 21-28

[10] D. MacKay, "Good error correcting codes based on very sparse matrices," *IEEE Trans. Infor. Theory*, vol. 45, no. 2, March 1999, pp. 399-431

[11] M. Moher, "An iterative multiuser decoder for near-capacity communications," *IEEE Trans. Comm.*, vol. 47, July 1998, pp. 870-880

[12] X. Wang and H. V. Poor, "Iterative (Turbo) soft interference cancellation and decoding for coded CDMA," *IEEE Trans. Comm.*, vol. 47, no. 7, July 1999, pp. 1046-1061

[13] P. D. Alexander, M. C. Reed, J. A. Asenstorfer, and C. B. Shlegel, "Iterative multiuser interference reduction: Turbo CDMA," *IEEE Trans. Comm.*, vol. 47, no. 7, July 1999, pp. 1008-1014

[14] B. Xia and W. E. Ryan, "Near-optimal convolutionally coded CDMA with iterative multiuser detection," *IEEE Globecom'01*, vol. 2, 2001, pp. 767-771

[15] V. V. Veeravalli and A. Mantravadi, "The coding-spreading tradoff in CDMA systems," *IEEE JSAC*, vol. 20, no. 2, Feb. 2002, pp.396-408

[16] L. Zhao, J. W. Mark, and Y. C. Yoon, "Coding-spreading tradeoff analysis for DS-CDMA systems," *IEEE VTC'01*, vol. 1, Fall 2001, pp.404-408

[17] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, John Wiley & Sons, New York, 1991

[18] S. Verdu, *Multiuser Detection*, Cambridge, UK: Cambridge Uni. Press, 1998

[19] R. R. Muller, "Combining multiuser detection with coding: promises and problems," Invited for 34th Annual Conference on Information Sciences and Systems, Princeton, NJ, USA, March 2000

[20] A. Kavcic, X. Ma, and M. Mitzenmacher, "Binary intersymbol interference channels: Gallager codes, density evolution and code performance bounds," submitted to *IEEE Trans. Infor. Theory*, Feb. 2001

[21] Stefan Weinzierl, Introduction to Monte Carlo methods, http://arxiv.org/abs/hep-ph/0006269

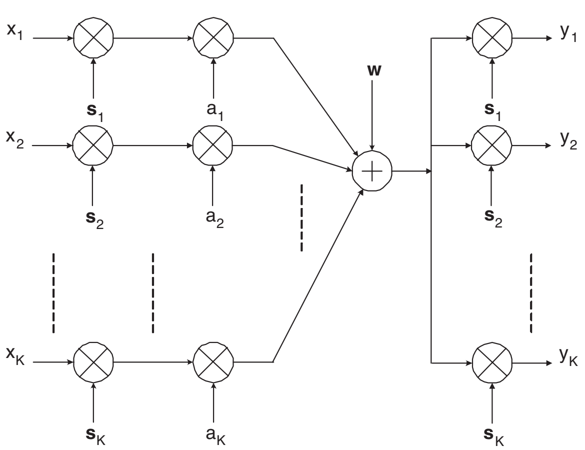

Fig. 1. Synchronous CDMA system model.

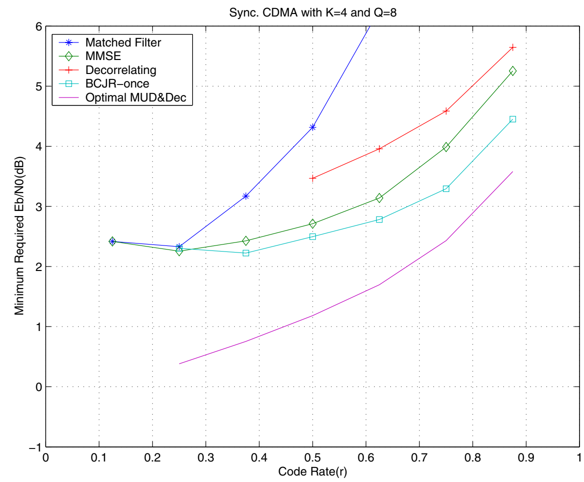

Fig. 2. The minimum required $\frac{E_b}{N_0}$ vs. code rate for a synchronous CDMA system with 4 users and bandwidth expansion factor 8.

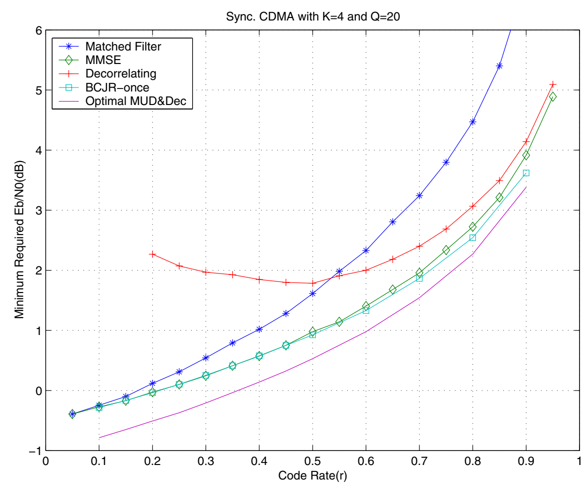

Fig. 3. The minimum required $\frac{E_b}{N_0}$ vs. code rate for a synchronous CDMA system with 4 users and bandwidth expansion factor 20.

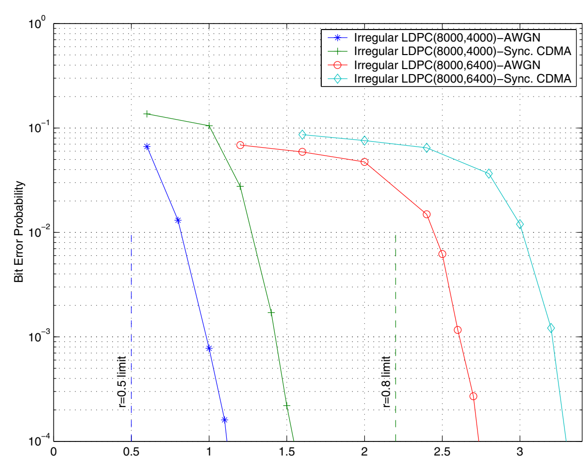

Fig. 4. The performance of one user in LDPC-coded synchronous CDMA systems with iterative multiuser detection/decoding. The systems have 4 users and bandwidth expansion factor 20.