Highest Unattainable Score in a 2-Region Dartboard Game

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I. BACKGROUND

While it may be true that unexpected 3.00 AM phone calls may be dreadful, unexpected 8.00 PM phone calls can be quite interesting. Here's one from an unexpected 8.00 PM phone call.

II. PROBLEM STATEMENT

Given a dartboard that has two distinct regions offering scores of P and Q, where P and Q are positive integers. If a dart lands on the board, the player gets a score of P or Q depending on the region in which the dart landed. If the dart lands outside the board, the player gets a 0 score. A player may throw as many darts as he/she chooses to. What is the highest positive number that cannot be a score in this game?

III. SOLUTION

Every possible score s obtained in the game may be represented in the form: $s = Px + Qy$, where x and y are the number of darts that landed in region having scores of P and Q respectively. The goal is to find the largest positive number that cannot be expressed in the above form.

A. The condition for the existence of a bounded solution

Let g denote the greatest common divisor (GCD) of P and Q. We can re-write P and Q as: $P = gP'$ and $Q = gQ'$. Then, any score that is attainable will be of the form as below.

$$\begin{aligned} s &= Px + Qy \\ &= g(P'x + Q'y) \end{aligned}$$

Thus, any attainable score will have g as a factor. Thus, if $g > 1$, any number that is not a multiple of g cannot be attained as a score. Therefore, in order to obtain all possible numbers above a certain threshold, it is necessary that $g = 1$. Or equivalently, the solution to the above problem is unbounded if $g > 1$.

Let l denote the least common multiple (LCM) of P and Q. Since GCD of P and Q is 1, the LCM of P and Q is PQ (since $l \times g = P \times Q$).

B. The highest unattainable score

Assume, without loss of generality, that P is the smaller of the two numbers. We divide all the non-negative integers in P groups, depending on the remainder left by the numbers when divided by P. Let Z_r contain the non-negative numbers which when divided by P will leave a remainder of r. From each group, we will first compute the smallest number that can be represented as $Px + Qy$ for some $x, y \geq 0$.

Let $r_i = iQ \bmod P$, the remainder obtained when $n = i \times Q$ is divided by P, where $i = 0, 1, \ldots P - 1$. Let $R = \{r_i | i = 0, 1, 2, \ldots P - 1\}$.

Lemma 1: R is a set containing some permutation of numbers from 0 through $P - 1$ with no repetition.

Proof: Since each r_i is constructed by using modulo operation with P, it is obvious that each r_i will range only from 0 through $P - 1$. By definition, the number of elements in R is P. Thus, we need to show that each element in R is unique, which we will prove by contradiction.

Assume that there exists two elements r_i and r_j, $0 \leq i, j < P$ and $i \neq j$, such that $r_i = r_j$. Without loss of generality, assume that $i < j$. Then, the positive number $(j - i)Q$ must be divisible by P (and Q obviously). As the highest possible value of j is $P - 1$ and the smallest possible value of i is 0,

$(j-i)Q < PQ$. Thus, $(j-i)Q$ is the least common multiple of P and Q, which is a contradiction. Thus, the elements of R must be unique. ■

Lemma 2: The smallest number in the set Z_{r_i} that can be expressed as $Px + Qy$ is iQ, where $i = 0, 1, \ldots P-1$, $r_i = iQ \bmod P$, and $x, y \geq 0$.

Proof: We will prove the above statement through contradiction. Assume that there exists another number $n < iQ$ in Z_{r_i} that can also be expressed as $Px + Qy$, for some $x, y \geq 0$. Observe that since $n < iQ$, $y < i$. In addition, we have:

$$iQ - n \quad = \quad -xP + (i-y)Q$$

Since iQ and n belong to the same group Z_{r_i}, $(iQ - n)$ is divisible by P. Therefore, from the above expression, $(i-y)Q$ must also be divisible by P. As the maximum value of i is $P-1$ and $0 \leq y \leq i$, $(y-i)Q < PQ$. Thus, $(y-i)Q$ is divisible by both P and Q and is less than PQ, a contradiction. ■

Let M_r denote the smallest number satisfying $(M_r \bmod P) = r$ that can be expressed in the form $Px + Qy$, $x, y \geq 0$.

Theorem: Given two positive numbers P and Q, whose GCD is 1, the largest positive number that cannot be expressed in the form $Px + Qy$, $x, y \geq 0$, is $PQ - P - Q$.

Proof: Consider the sequence of remainders in R. $(P-1)Q$ is the smallest number that can be represented in the desired form among all numbers that will leave a remainder of r_{P-1} when divided by P. Thus, any number higher than or equal to $(P-1)Q$ may be expressed in the desired from. Thus, the highest number that cannot be expressed in the desired from must be less than $(P-1)Q$.

Let $r_j = [(P-1)Q - j] \bmod P$, where $j \in \{0, 1, 2, \ldots P-1\}$. Observe that $M_{r_j} \leq (P-1)Q - j, \forall j \in \{0, 1, 2, \ldots P-1\}$. Thus, the highest positive number that cannot be expressed in the desired from is $(P-1)Q - P = PQ - P - Q$. ■